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Abstract

The notion of multivariate spacings was introduced and studied by Deheuvels, P. (1983) for data uniformly distributed on the unit cube. Later on, Janson, S. (1987) extended the results to bounded sets, and obtained a very fine result, namely, he derived the exact asymptotic distribution of the maximal spacing. These results have been very useful in many statistical applications.

We extend Janson's result to the case where the data are generated from a positive, bounded support Lipchitz continuous density function, and develop a convexity test for the support of a distribution.

Keywords: maximal spacing; convexity test; non-parametric density estimation

1 Introduction

The notion of spacings, which for one dimensional data are just the differences between two consecutive order statistics, have been extensively studied in the one dimensional setting; see e.g., the review papers by Pyke, R. (1965, 1972). Many important applications for testing and estimation problems, are derived from the study of the asymptotic behavior of the spacings. Applications for testing problems dates back to Proschan, F. and Pyke, R. (1967) who address the asymptotic theory of a class of tests for Increasing Failure Rate. For estimation problems, Ranneby, B (1984) propose the maximum spacing estimation method to estimate the parameters of a univariate statistical model.

Particular attention has been devoted to the behavior of the maximal (largest) spacing (see for instance Stevens, W. L. (1939), Devroye, L. (1981) and Deheuvels, P. (1983)).

For points that are uniformly distributed in the unit cube $K = [0, 1]^d$, Deheuvels, P. (1983) introduced the notion of maximal spacing for the multivariate setting as the volume of the largest cube C, parallel to the unit cube, that is contained in $[0, 1]^d$ and do not contain any of the n sample points.

Janson, S. (1987) extended these results for a sample of random vectors uniformly distributed on a bounded set $S \subset \mathbb{R}^d$ such that $|S| = 1$ (where $|.|$ denotes the Lebesgue measure on $\mathbb{R}^d$), and the cube C is replaced by any fixed bounded convex set with a nonempty interior. In addition to extending results on strong bounds, he derived the exact asymptotic distribution of the maximal spacing (see Theorems 1 and Corollary 1). The notion of maximal multivariate spacing and, in particular, Janson's result, have been used to solve different statistical problems. In set estimation (see, for instance, Cuevas, A. and Fraiman, R. (1997) and Cuevas, A. and Rodriguez-Casal, A. (2004)), it is used to prove the optimality of the rates of convergence.

We seek to achieve the following:

i) We extend Janson's result to the case where the data are generated by a Lipchitz continuous density function with bounded support S, which is bounded from below by a positive constant on S. This will require us to extend the notion of maximal spacing to the case of non-uniform data.

ii) Based on the previous result, we develop a convexity test for the support S and compare it with some recent results presented by Delicado, P.; Hernández, A. and Lugosi, G. (2014).

The paper is organized as follows. First, we introduce the new notion of maximal spacing and state the asymptotic results for the maximal spacing. Next, in Section 3, we address the convexity test problem for two different settings: the semi-parametric case (where the set is unknown, but the data are uniform) and the nonparametric case (where the data are generated by an unknown density f). We study the asymptotic behavior of the tests for both settings and conduct a small simulation study. Finally, as we show in the Appendix, the asymptotic distribution of the maximal spacing is derived in three steps. We start with a density that is a mixture of uniform laws with disjoint supports, then consider a density that is a uniform mixture and finally consider a density that is Lipchitz continuous and bounded from below.

2 Main definitions and results

We first introduce notation that will be used throughout the manuscript. Given a set S, we denote by ∂S, $\mathring{S}$, and $\overline{S}$ the boundary, interior and closure of S, respectively. We denote by $\mathcal{B}(x, \varepsilon)$ the closed ball of radii ε centered at x and by $\omega_d = |\mathcal{B}(x, 1)|$ the Lebesgue measure of the unit ball in $\mathbb{R}^d$. Given $\lambda \in \mathbb{R}$, $A, C \subset \mathbb{R}^d$ we denote $\lambda A = \{\lambda a : a \in A\}$, $A \oplus C = \{a + c : a \in A, c \in C\}$, and $A \ominus C = \{x : \{x\} \oplus C \subset A\}$. For the sake of simplicity, we use the notation x+C, instead of $\{x\} \oplus C$. If $\lambda \geq 0$ we denote $A^\lambda = A \oplus \lambda\mathcal{B}(0, 1)$, and $A^{-\lambda} = A \ominus \lambda\mathcal{B}(0, 1)$. Given $A, C \subset \mathbb{R}^d$ two non-empty compact sets, the Hausdorff (or Pompeiu-Hausdorff) distance between them is given by

$$d_H(A, C) = \max\left\{\max_{a \in A} d(a, C),\ \max_{c \in C} d(c, A)\right\}$$

where $d(a, C) = \inf\{\|a - c\| : c \in C\}$. Given a set $S \subset \mathbb{R}^d$, we denote by $\mathcal{H}(S)$ the convex hull of S (that is, the minimal convex set that contains S).

Let $S \subset \mathbb{R}^d$ be a bounded set with Lebesgue measure 1, but with Lebesgue measure zero of its boundary ∂S. Let $\aleph_n = \{X_1, \dots, X_n\}$ be iid random vectors uniformly distributed on S, and A a bounded convex set. According to Janson (Janson, S. (1987)), the maximal spacing is defined as:

$$\Delta^*(\aleph_n) = \sup\Big\{r:\ \exists x \text{ such that } x + rA \subset S \setminus \aleph_n\Big\}.$$

To generalize Janson's result to the non-uniform case, we need to extend the maximal spacing definition. When the sample is drawn according to a probability measure P_X, we consider the probability measure of the largest empty λA set. When $|A| = 1$ $P_X(x + \lambda A) \sim f(x)\lambda^d$ for sufficiently small λ. This leads us to define the maximal spacing extension as follows:

Definition 1. *Let $\aleph_n = \{X_1, \dots, X_n\}$ be an iid random sample of points in $\mathbb{R}^d$, drawn according to a density f with bounded support S, and let $A \subset \mathbb{R}^d$ be a convex and compact set such that $|A| = 1$ (where $|\cdot|$ denote the Lebesgue measure) and its barycentre is the origin of $\mathbb{R}^d$. We define:*

$$\Delta(\aleph_n) = \sup\Big\{r:\ \exists x \textit{ such that } x + \frac{r}{f(x)^{1/d}}A \subset S \setminus \aleph_n\Big\}, \tag{1}$$

$$V(\aleph_n) = \Delta^d(\aleph_n),$$

and

$$U(\aleph_n) = n\Delta^d(\aleph_n) - \log(n) - (d-1)\log\big(\log(n)\big) - \log(\alpha_A),$$

where $\alpha_A > 0$ is the constant defined in Janson, S. (1986). For instance, if A is a cube, $\alpha_A = 1$; if A is a ball, then $\alpha_A = \frac{1}{d!}\left(\frac{\sqrt{\pi}\Gamma(\frac{d}{2}+1)}{\Gamma(\frac{d+1}{2})}\right)^{d-1}$.

Finally, we denote U, a random variable, such that $\mathbb{P}(U \leq t) = \exp\big(-\exp(-t)\big)$.

When $|S| = 1$ and the sample is uniformly drawn on S, the following result can be found in Janson, S. (1987), Theorem 1:

Theorem 1. *Let $S \subset \mathbb{R}^d$ be a bounded set such that $|S| = 1$ and $|\partial S| = 0$. Let $\aleph_n = \{X_1, \dots, X_n\}$ and X be iid random vectors uniformly distributed on S: then,*

$$U(\aleph_n) \xrightarrow{\mathcal{L}} U \quad \textit{when } n \to \infty.$$

A simple rescaling extends this result to the case where $|S| \neq 1$:

Corollary 1. *Let $S \subset \mathbb{R}^d$ be a bounded set such that $|\partial S| = 0$ and $|S| > 0$. Let $\aleph_n = \{X_1, \dots, X_n\}$ and X be iid random vectors uniformly distributed on S; then,*

$$U(\aleph_n) \xrightarrow{\mathcal{L}} U \quad \textit{when } n \to \infty.$$

We are interested in the asymptotic behavior of $U(\aleph_n)$ as $n \to \infty$, when the density is not uniform. The main result (see Theorem 2 below) is presented for Lipschitz continuous densities.

Theorem 2. *Let f be a density with compact support $S \subset \mathbb{R}^d$. Suppose that f is Lipschitz (with constant K) and that there exist positive constants f_0, f_1 such that for all $x \in S$, $0 < f_0 \leq f(x) \leq f_1$. Then, we have the following:*

$$U(\aleph_n) \xrightarrow{\mathcal{L}} U \qquad \textit{when } n \to \infty.$$

$$\liminf_{n\to+\infty} \frac{nV(\aleph_n) - \log(n)}{\log(\log(n))} = d - 1 \ \textit{a.s.}$$

$$\limsup_{n\to+\infty} \frac{nV(\aleph_n) - \log(n)}{\log(\log(n))} = d + 1 \ \textit{a.s.}$$

The proof is given in the Appendix.

3 A new test for convexity

3.1 The semi-parametric case

In this section, we propose, using the concept of maximal spacing defined in Section 2, a consistent hypothesis test, based on an iid sample $\{X_1, \ldots, X_n\}$ uniformly distributed on a compact set S, to decide whether S is convex or not. The main idea is that, if the set is not convex the maximal spacing between the convex hull of the set and the sample will not converge to zero. Because the set is unknown, instead of the convex hull of the set, we consider the convex hull of the sample. To change the set by its convex hull, we prove some previous results which guarantee that the maximal spacings will be close.

Definition 2. *Let $S \subset \mathbb{R}^d$ be a bounded set satisfying $\mathring{S} \neq \emptyset$. We define the maximal spacing of S (denoted $\Delta(S)$) as*

$$\Delta(S) = \sup\big\{r : \exists x \in S \textit{ such that } \mathcal{B}(x,r) \subset S\big\}.$$

Although there is an abuse of notation here, it is important to note that to define $\Delta(S)$, we do not need a sample or density. In that sense, it is different from the one defined in 1. Moreover, although the set $\aleph_n$ is bounded, the condition $\mathring{\aleph}_n \neq \emptyset$ is not satisfied.

Proposition 1. *Let A and B be bounded and nonempty subsets of $\mathbb{R}^d$. If for some $\varepsilon > 0$, $d_H(A,B) \leq \varepsilon$ and $d_H(\partial A, \partial B) \leq \varepsilon$, then*

$$\big|\Delta(A) - \Delta(B)\big| \leq 2\varepsilon.$$

Proof. It is enough to prove that:

1. $\big\{x \in A : d(x, \partial A) > 2\varepsilon\big\} \subset B$
2. $\big\{x \in B : d(x, \partial B) > 2\varepsilon\big\} \subset A$

From the first inclusion, we obtain that $\Delta(B) \geq \Delta(A) - 2\varepsilon$, while from the second, we obtain that $\Delta(A) \geq \Delta(B) - 2\varepsilon$. Then, $\big|\Delta(A) - \Delta(B)\big| \leq 2\varepsilon$. To obtain the first inclusion (the second one is analogous), we suppose that there exists $x \in A$ such that $d(x, \partial A) > 2\varepsilon$ but $x \notin B$. Because $A \subset B^\varepsilon$, we have $x \in B^\varepsilon \setminus B$. Then $d(x, \partial B) \leq \varepsilon$, which implies that

$$d(x, \partial A) = d_H(\{x\}, \partial A) \leq d_H(\{x\}, \partial B) + d_H(\partial B, \partial A) \leq 2\varepsilon.$$

□

The following proposition shows that if the set S is not convex, then the maximal spacing of the set $\mathcal{H}(S) \setminus S$ is strictly positive.

Proposition 2. *Let $S \subset \mathbb{R}^d$ be a non–convex, compact non-empty set, such that $S = \overline{\mathring{S}}$. Then,*

$$0 < \Delta\big(\mathcal{H}(S) \setminus S\big).$$

Proof. We first prove that:

$$\mathcal{H}(S) = \overline{\mathcal{H}(\mathring{S})}. \tag{2}$$

The fact that $\mathcal{H}(S)$ is a closed set implies that $\overline{\mathcal{H}(\mathring{S})} \subset \mathcal{H}(S)$. Thus, (2) will hold if we prove that $\mathcal{H}(S) \subset \overline{\mathcal{H}(\mathring{S})}$. However, $S = \overline{\mathring{S}} \subset \mathcal{H}(S)$ and $\overline{\mathring{S}} \subset \overline{\mathcal{H}(\mathring{S})}$ entail that (2) follows from $\overline{\mathcal{H}(\mathring{S})}$ being a convex set. Because S is not convex, there exist $x, y \in S$ such that the segment $[x, y]$ joining them is not contained in S. However, $\mathcal{H}(S)$ is convex, and therefore the segment is contained in $\mathcal{H}(S)$. Because S is compact, there exist $\delta > 0$ and $t \in [x, y]$ such that $\mathcal{B}(t, \delta) \cap S = \emptyset$. By (2), we have that $\mathcal{B}(t, \delta) \cap \mathcal{H}(\mathring{S}) \subset \mathcal{H}(S) \setminus S$, and therefore, $\Delta\big(\mathcal{H}(S) \setminus S\big) \geq \Delta\big(\mathcal{B}(t, \delta) \cap \mathcal{H}(\mathring{S}) \setminus S\big) > 0$. □

If $S \subset \mathbb{R}^d$ is convex and $\aleph_n = \{X_1, \ldots, X_n\}$ is an iid random sample, uniformly drawn on S, Walther, G. (1996) proved that

$$d_H\big(S, \mathcal{H}(\aleph_n)\big) = \mathcal{O}\Big(\big(\log(n)/n\big)^{1/d}\Big).$$

Moreover, under an additional regularity condition on ∂S, it has also been proven in Walther, G. (1996) that the previous order can be improved. More specifically, the following results holds

$$d_H\big(S, \mathcal{H}(\aleph_n)\big) = \mathcal{O}\Big(\big(\log(n)/n\big)^{2/(d+1)}\Big).$$

The regularity condition is the following:

Condition (P): For all $x \in \partial S$ there exists a unique vector $\xi = \xi(x)$ with $\|\xi\| = 1$, such that $\langle y, \xi\rangle \leq \langle x, \xi\rangle$ for all $y \in S$, and

$$\|\xi(x) - \xi(y)\| \leq l\|x - y\| \quad \forall\ x, y \in \partial S,$$

where l is a constant. We will denote by $\mathcal{A}_P$ the class of convex subsets that satisfy condition **(P)**.

Theorem 3. *Let $S \subset \mathbb{R}^d$ be a compact subset such that $S = \overline{\mathring{S}}$. For the following decision problem*

$$\begin{cases} H_0: & \text{the set } S \text{ is convex} \\ H_1: & \text{the set } S \text{ is not convex,} \end{cases} \tag{3}$$

the test based on the statistic $\tilde{V}_n = \omega_d \Delta^d\big(\mathcal{H}(\aleph_n) \setminus \aleph_n\big)$ with the critical region given by

$$RC = \big\{\tilde{V}_n > c_{n,\gamma}\big\},$$

where

$$c_{n,\gamma} = \frac{|\mathcal{H}(\aleph_n)|}{n}\Big(-\log\big(-\log(1-\gamma)\big)+\log(n)+(d-1)\log\big(\log(n)\big)+\log(\alpha)\Big) = \mathcal{O}\big(\log(n)/n\big),$$

and α is the constant defined in (1), is asymptotically of level smaller or equal to γ. Moreover if $S \in \mathcal{A}_P$ the asymptotic level equals γ. If S is not convex, the test has power one for n sufficiently large n.

Proof. First observe that, if S is convex (not necessarily in $\mathcal{A}_P$) from Theorem 1 and the following inequality

$$\Delta\big(\mathcal{H}(\aleph_n) \setminus \aleph_n\big) \leq \Delta\big(S \setminus \aleph_n\big),$$

we obtain that $\mathbb{P}\big(\Delta\big(\mathcal{H}(\aleph_n) \setminus \aleph_n\big) > c_{n,\gamma}\big) \leq \gamma$.

We now prove that for the case $S \in \mathcal{A}_P$, $\mathbb{P}_{H_0}\big(\tilde{V}_n > c_{n,\gamma}\big) \to \gamma$ and, for n sufficiently large n, $\mathbb{P}_{H_1}\big(\tilde{V}_n > c_{n,\gamma}\big) = 1$. First observe that under H_0, $\aleph_n \subset \mathcal{H}(\aleph_n) \subset S$ for all $n > 0$. As condition **(P)** is satisfied we know that

$$d_H\big(\mathcal{H}(\aleph_n), S\big) = \mathcal{O}\Big(\big(\log(n)/n\big)^{2/(d+1)}\Big), \tag{4}$$

which implies that

$$d_H\big(\partial\mathcal{H}(\aleph_n), \partial S\big) = \mathcal{O}\Big(\big(\log(n)/n\big)^{2/(d+1)}\Big). \tag{5}$$

We assume that $|S|$ is known. Indeed, by (4) and (5) together with Theorem 2 in Cuevas, A., Fraiman, R. and Pateiro-Lopez, B. (2012), we have that $|\mathcal{H}(\aleph_n)| \to |S|$. Thus, we use $c'_{n,\gamma} = \frac{|S|}{\mathcal{H}(\aleph_n)} c_{n,\gamma}$ instead of $c_{n,\gamma}$.

By Proposition 1, we have that under H_0:

$$\Big|\Delta\big(\mathcal{H}(\aleph_n) \setminus \aleph_n\big) - \Delta\big(S \setminus \aleph_n\big)\Big| = \mathcal{O}\Big(\big(\log(n)/n\big)^{2/(d+1)}\Big).$$

If we denote $\varepsilon_n = \Big|\Delta\big(\mathcal{H}(\aleph_n) \setminus \aleph_n\big) - \Delta\big(S \setminus \aleph_n\big)\Big|$ we can derive that

$$\Delta^d\big(\mathcal{H}(\aleph_n) \setminus \aleph_n\big) = \Delta^d(S \setminus \aleph_n) + d\varepsilon_n \Delta^{d-1}(S \setminus \aleph_n) + o\big(\varepsilon_n \Delta^{d-1}(S \setminus \aleph_n)\big).$$

Applying Lemma 5 given in Subsection 4.1.2, we consider $a > 0$ such that

$$V_n - \tilde{V}_n = \mathcal{O}\big((\log(n)/n)^{1+a}\big),$$

where $V_n = \omega_d \Delta^d(S \setminus \aleph_n)$, and

$$\gamma_n = \mathbb{P}_{H_0}\big(\tilde{V}_n > c'_{n,\gamma}\big) = \mathbb{P}_{H_0}\big((\tilde{V}_n - V_n) + V_n > c'_{n,\gamma}\big) = \mathbb{P}_{H_0}\big(V_n > c'_{n,\gamma} + o(c'_{n,\gamma})\big).$$

Therefore, by Theorem 1, $\gamma_n \to \gamma$.
To prove that for sufficiently large n the power is 1 if S is not convex, we use Proposition 2, but instead of S, we use $\aleph_n$ and instead of $\mathcal{H}(\aleph_n)$, we use $\mathcal{H}(S)$. To do so, we first observe the following

$$\mathcal{H}(S)^{-2\varepsilon_n} \subset \mathcal{H}(\aleph_n) \subset \mathcal{H}(S) \quad a.s., \tag{6}$$

where $\varepsilon_n = d_H(S, \aleph_n)$. The second inclusion is immediate. To prove the first one, we proceed by contradiction: suppose that there exists $x \in \mathcal{H}(S)^{-\varepsilon_n}$ but $x \notin \mathcal{H}(\aleph_n)$. Because $x \notin \mathcal{H}(\aleph_n)$, there exists a halfspace H_x such that $x \in H_x$ and $H_x \cap \aleph_n = \emptyset$. Now, we take $z \in \mathcal{B}(x, 2\varepsilon_n) \cap H_x$ such that $\mathcal{B}(z, \varepsilon_n) \subset H_x$. Because $z \in \mathcal{B}(x, \varepsilon_n) \subset \mathcal{H}(S)$, the halfspace H_z parallel to H_x such that $z \in \partial H_z$ meets S at some point s. Then $\mathcal{B}(s, \varepsilon_n) \subset H_x$, while $\varepsilon_n = d_H(S, \aleph_n)$, which implies that there exists $X_i \in \mathcal{B}(s, \varepsilon_n)$, contradicting $H_x \cap \aleph_n = \emptyset$. Then, we have the following:

$$\begin{aligned}
\Delta\big(\mathcal{H}(S) \setminus S\big) - 2\varepsilon_n &\leq \Delta\big(\mathcal{H}(S)^{-2\varepsilon_n} \setminus S\big) \\
&\leq \Delta\big(S^{-2\varepsilon_n} \setminus \aleph_n\big) \\
&\leq \Delta\big(\mathcal{H}(\aleph_n) \setminus \aleph_n\big).
\end{aligned} \tag{7}$$

Because $\mathcal{H}(S) < \infty$, we have that by (6), $c'_{n,\gamma} \to 0$. Then $\mathbb{P}_{H_1}(\tilde{V}_n > c_{n,\gamma}) = 1$ a.s. for sufficiently large n.

□

3.2 The non-parametric case

We now assume that we have a sample $\aleph_n = \{X_1, \ldots, X_n\}$ of iid random vectors in $\mathbb{R}^d$ drawn according to an unknown density f. We propose to plug in a density estimator $\hat{f}_n$ on (1), compute

$$\hat{\delta}\big(\mathcal{H}(\aleph_n) \setminus \aleph_n\big) = \sup\left\{r : \ \exists x \text{ such that } x + \frac{r}{\hat{f}_n(x)^{1/d}} A \subset \mathcal{H}(\aleph_n) \setminus \aleph_n\right\},$$

and reject H_0 (the support is convex) if $\hat{\delta}(\mathcal{H}(\aleph_n) \setminus \aleph_n)$ is sufficiently large.

To increase the power of our test we need to find a density estimator that overestimates the density when the support is not convex. To do so, we propose the following density estimator.

Definition 3. *Let Vor*(X_i) *be the Voronoi cell of the point* X_i *i.e. Vor*$(X_i) = \{x : \|x - X_i\| = \min_{y \in \aleph_n} \|x - y\|\}$. *If* K *is a kernel function and* $\tilde{f}_{h_n}(x) = \frac{1}{nh_n^d} \sum K((x - X_i)/h_n)$ *denotes the usual kernel density estimator, we define:*

$$\hat{f}_n(x) = \max_{i, x \in Vor(X_i)} \tilde{f}(X_i). \tag{8}$$

For the uniform case, we require that the boundary of the support be smooth enough to derive the asymptotic behavior. In this more general setup, we will not have a convergent level estimation and will only have a level majorization (the price to pay to estimate the density).

Condition (P'): For a given kernel function K, we say that S is standard with respect to K and with respect to the Lebsegue measure if there exist positive constants r_0, c_S and c_K such that for all $x \in S$, $\int_{u \in S} K((u-x)/r)du \geq c_K r^d$ and $|\mathcal{B}(x, r) \cap S| \geq c_S r^d$. We denote $\mathcal{C}_K$ as the class of convex sets that satisfy condition **(P')** and $\mathcal{A}_K$ as the class of all the sets that satisfy condition **(P')**.

We require the following assumptions on the kernel:

Definition 4. *Let* $\mathcal{K}$ *be the set of positive kernel functions such that* $\int \|u\| K(u)du < \infty$ *and* $K(u) = \phi(p(u))$, *where* p *is a polynomial and* ϕ *a is bounded real function of bounded variation.*

Notice that all the usual kernels are in $\mathcal{K}$. Sometimes, we require the following condition on the underlying density f.

Condition (B): A density f with support S fulfills condition B if it is Lipschitz continuous and if there exists $f_0 > 0$ such that $f(x) \geq f_0$ for all $x \in S$.

Theorem 4. *Let* $K \in \mathcal{K}$ *and* $\hat{f}_n$ *be defined as in Definition 3. Assume that* $h_n = O(n^{-\beta})$ *for some* $0 < \beta < 1/d$. *We also assume that the unknown density fulfills condition B. For the following decision problem,*

$$\begin{cases} H_0: & S \in \mathcal{C}_K \\ H_1: & S \notin \mathcal{C}_K, \end{cases} \tag{9}$$

the test based on the statistic $\tilde{V}_n = \delta\big(\mathcal{H}(\aleph_n) \setminus \aleph_n\big)$ *with critical region* $RC = \{\tilde{V}_n \geq C_{n,\gamma}\}$, *where*

$$C_{n,\gamma} = \frac{1}{n}\Big(-\log(-\log(1-\gamma)) + \log(n) + (d-1)\log(\log(n)) + \log(\alpha)\Big),$$

has an asymptotical level smaller than γ. *Moreover, if* $S \in \mathcal{A}_K$ *is not convex, the power is 1 for sufficiently large* n.

Remark 1. *Condition B seems to be restrictive, however is unavoidable. Indeed, we quote from Delicado, P.; Hernández, A. and Lugosi, G. (2014): "...it is impossible (in a*

well-defined sense described below) to design a decision rule that behaves asymptotically correctly for all bounded densities of bounded support. This shows that an assumption like the density being bounded away from zero on its support is necessary for consistent decision rules." (see Theorem 2).
However, condition P' is satisfied for a large class of kernel functions, because convex sets are standard.

The proof of this theorem is given in the next subsection. To do so, we prove Propositions 3, 4, 5 and 6.

3.2.1 More results on the test and proofs

In the first subsection, we assume that the proposed density estimator fulfills some "good conditions", and in the second we prove that the density defined in (8) fulfills those conditions when the support belongs to $\mathcal{A}_{\mathcal{K}}$.

Asymptotic properties of the test

Proposition 3. *Assume that the unknown density f fulfils condition (B). We suppose that $\hat{f}_n$ is a density estimator that fulfills the following:*

(i) There exists a sequence $\varepsilon_n^+ \to 0$ such that for all $x \in S$, $\left(\frac{f(x)}{\hat{f}_n(x)}\right)^{1/d} \geq 1 - \varepsilon_n^+$.

(ii) There exists a sequence $\varepsilon_n^- \to 0$ and a constant $\lambda_0 > 0$ such that for all $x \in \mathcal{H}(\aleph_n)$, $(\hat{f}_n(x))^{1/d} \geq \lambda_0 - \varepsilon_n^-$.

If we consider the following decision problem

$$\begin{cases} H_0: & S \text{ is convex} \\ H_1: & S \text{ is not convex} \end{cases} \tag{10}$$

the test based on the statistic $\hat{\delta}\big(\mathcal{H}(\aleph_n) \setminus \aleph_n\big)$ with critical region

$$RC = \left\{ \hat{\delta}\big(\mathcal{H}(\aleph_n) \setminus \aleph_n\big) > \Big(\lambda \frac{\log(n)}{n}\Big)^{1/d} \right\}$$

is asymptotically consistent if λ sufficiently large.

Proof. **When the support is convex:** We prove the following:

$$\Delta(\aleph_n) \geq (1 - \varepsilon_n^+)\hat{\delta}\big(\mathcal{H}(\aleph_n) \setminus \aleph_n\big). \tag{11}$$

Observe that $\hat{\delta}(\mathcal{H}(\aleph_n) \setminus \aleph_n) = t \Rightarrow \; \exists x$ such that $\mathring{\mathcal{B}}\big(x, t/(\hat{f}_n(x))^{1/d}\big) \subset \mathcal{H}(\aleph_n) \setminus \aleph_n$. Because S is convex, $\mathcal{H}(\aleph_n) \subset S$. Then, $\hat{\delta}(\mathcal{H}(\aleph_n) \setminus \aleph_n) = t \Rightarrow \exists x$ such that $\mathring{\mathcal{B}}\big(x, t/(\hat{f}_n(x))^{1/d}\big) \subset S \setminus \aleph_n$. From the equality

$$\left(\frac{f(x)}{\hat{f}_n(x)}\right)^{1/d} \mathring{\mathcal{B}}\left(x, \frac{t}{f(x)^{1/d}}\right) = \mathring{\mathcal{B}}\left(x, \frac{t}{\hat{f}_n(x)^{1/d}}\right),$$

we derive the following: if $\hat{\delta}(\mathcal{H}(\aleph_n) \setminus \aleph_n) = t$, then there exists x such that , $(1 - \varepsilon_n^+)\mathring{\mathcal{B}}\big(x, t/(f(x))^{1/d}\big) \subset S \setminus \aleph_n$. Therefore, we have $\hat{\delta}(\mathcal{H}(\aleph_n) \setminus \aleph_n) = t \Rightarrow \Delta(\aleph_n) \geq (1 - \varepsilon_n^+)t$. According to Lemma 5 we have $\hat{\delta}(\mathcal{H}(\aleph_n) \setminus \aleph_n) \leq \frac{r_f}{1-\varepsilon_n^+} \left(\frac{\log(n)}{n}\right)^{1/d}$ eventually almost surely.

When the support is not convex:
By assumption (ii), we know that

$$\hat{\delta}\big(\mathcal{H}(\aleph_n) \setminus \aleph_n\big) \geq (\lambda_0 - \varepsilon_n^-)\Delta\big(\mathcal{H}(\aleph_n) \setminus \aleph_n\big),$$

and from (7), we obtain

$$\hat{\delta}\big(\mathcal{H}(\aleph_n) \setminus \aleph_n\big) \geq (\lambda_0 - \varepsilon_n^-)\big(r_0 - 2d_H(S, \aleph_n)\big),$$

where $r_0 = \Delta\big(\mathcal{H}(S) \setminus S\big)$. Because we are assuming that S is not convex we have, by Proposition 2, that $r_0 > 0$. Applying Lemma 5, we have $d_H(S, \aleph_n) \leq r_f(\log(n)/n)^{1/d}$ eventually almost surely. Thus, when the support is not convex

$$\hat{\delta}\big(\mathcal{H}(\aleph_n) \setminus \aleph_n\big) \geq r_0\lambda_0 + o(1) \text{ (eventually almost surely).} \tag{12}$$

□

Proposition 4. *Assume that the unknown density f fulfills condition (B). Suppose that the density estimator $\hat{f}_n$ satisfies the following conditions:*

There exists a sequence ε_n^+ such that $\log(n)\varepsilon_n^+ \to 0$ and for all $x \in S$, $\left(\frac{f(x)}{\hat{f}_n(x)}\right)^{1/d} \geq 1 - \varepsilon_n^+$. If we consider the decision problem (10), based on the test statistic $\tilde{V}_n = \hat{\delta}^d\big(\mathcal{H}(\aleph_n) \setminus \aleph_n\big)$ with the critical region $RC = \{\tilde{V}_n \geq C_{n,\gamma}\}$, where

$$C_{n,\gamma} = \frac{1}{n}\big(- \log(-\log(1 - \gamma)) + \log(n) + (d - 1)\log(\log(n)) + \log(\alpha)\big),$$

the level is asymptotically smaller than γ.

Proof. By (11), we have

$$\mathbb{P}(\tilde{V}_n \geq C_{n,\gamma}) \leq \mathbb{P}\big(V(\aleph_n) \geq (1 - \varepsilon_n^+)C_{n,\gamma}\big).$$

Then, by Corollary 1, it follows that $\mathbb{P}(\tilde{V}_n \geq C_{n,\gamma})$ is bounded from above by

$$\mathbb{P}\Big(U \geq -(1-\varepsilon_n^+)^d \log\big(-\log(1-\gamma)\big) + \big((1-\varepsilon_n^+)^d - 1\big)\big(\log(n) + (d-1)\log(\log(n)) + \log(\alpha)\big)\Big).$$

Finally, because $\log(n)\varepsilon_n^+ \to 0$, we obtain:

$$\mathbb{P}(\tilde{V}_n \geq C_{n,\gamma}) \leq \mathbb{P}\big(U \geq -\log(-\log(1 - \gamma)) + o(1)\big) \to \gamma.$$

□

Proposition 5. *Assume that the unknown density f fulfills condition (B). Suppose that the density estimator $\hat{f}_n$ satisfies that there exist a sequence $\varepsilon_n^- \to 0$ and a constant $\lambda_0 > 0$ such that, for all $x \in \mathcal{H}(\aleph_n)$, $(\hat{f}_n(x))^{1/d} \geq \lambda_0 - \varepsilon_n^-$.*
If we consider the decision problem (10), along with the test statistic $\tilde{V}_n = \hat{\delta}^d\big(\mathcal{H}(\aleph_n)\backslash\aleph_n\big)$ with critical region $RC = \{\tilde{V}_n \geq C_{n,\gamma}\}$ where :

$$C_{n,\gamma} = \frac{1}{n}\big(- \log(-\log(1-\gamma)) + \log(n) + (d-1)\log(\log(n)) + \log(\alpha)\big),$$

if S is not convex, the power of the test is 1 for sufficiently large n.

Proof. It is clear that for $\gamma = \gamma_0$, $C_{n,\gamma} = O(\log n/n)$. However, (12) entails that $\tilde{V}_n \geq \lambda_0^d \delta^d(\mathcal{H}(S) \setminus S) + o(1)$ (with $\delta^d(\mathcal{H}(S) \setminus S) > 0$) eventually almost surely. □

An appropriate density estimator To prove Theorem 4, we have to show that the density estimator introduced in Definition 3 fulfills conditions (i) and (ii) of Proposition 3. We show in the next Proposition 6 that these conditions hold.

Proposition 6. *Assume that the unknown density f fulfills condition (B). We suppose that $K \in \mathcal{K}$ and that h_n satisfies:*

a) $h_n \log(n) \to 0$, $\quad \frac{nh_n^d}{\big(\log(n)\big)^2 \big|\log(h_n)\big|} \to \infty$, $\quad \frac{\log(h_n)}{\log(\log(n))} \to \infty$, *and*

b) there exists a constant $c > 0$ such that $h_n^d \leq c h_{2n}^d$.

(Note that for all $\beta \in (0, 1/d)$, $h_n = h_0 n^{-\beta}$ fulfills all these conditions).

Let $\hat{f}_n(x)$ be the density estimator introduced in Definition 3. Then,

(i) there exists a sequence $\varepsilon_n^+ \to 0$ such that for all $x \in S$, $\left(\frac{f(x)}{\hat{f}_n(x)}\right)^{1/d} \geq 1 - \varepsilon_n^+$, and

(ii) there exist a sequence $\varepsilon_n^- \to 0$ and a constant $\lambda_0 > 0$ such that for all $x \in \mathcal{H}(\aleph_n)$, $(\hat{f}_n(x))^{1/d} \geq \lambda_0 - \varepsilon_n^-$.

Proof. We start the proof of (i). We first write that:

$$\max_{x \in S} \big(\hat{f}_n(x) - f(x)\big) \leq \max_{x \in S} \big|\hat{f}_n(x) - \mathbb{E}\hat{f}_n(x)\big| + \max_{x \in S} \big(\mathbb{E}\hat{f}_n(x) - f(x)\big).$$

By Theorem 2.3 in Giné, E. and Guillou, A. (2002), there exists a constant C_1 such that:

$$\sqrt{\frac{nh_n^d}{-\log(h_n)}} \sup_{x \in \mathbb{R}^d} \big|\tilde{f}_n(x) - \mathbb{E}\tilde{f}_n(x)\big| \leq C_1 \text{ a.s.}$$

Thus

$$\sqrt{\frac{nh_n^d}{-\log(h_n)}} \sup_{x \in \aleph_n} \big|\tilde{f}_n(x) - \mathbb{E}\tilde{f}_n(x)\big| \leq C_1 \text{ a.s.},$$

and therefore,

$$\sqrt{\frac{nh_n^d}{-\log(h_n)}}\sup_{x\in\aleph_n}\left|\hat{f}_n(x)-\mathbb{E}\hat{f}_n(x)\right|\leq C_1 \text{ a.s.} \tag{13}$$

The proof of (i) will be complete if we find a proper bound for $\max_{x\in S}(\mathbb{E}\hat{f}_n(x)-f(x))$. We first note that the standardness assumption (with respect to the Lebesgue measure) ensures that there exists a constant r_S such that for all $X_i \in \aleph_n$, for all $x \in \text{Vor}(X_i)\cap S$, $\|x-X_i\| \leq d_H(S,\aleph_n) = r_S(\log(n)/n)^{1/d}$, where for the last equality, we used Theorem 4 in Cuevas, A. and Rodriguez-Casal, A. (2004) and the fact that S is standard. We denote $\rho_n = r_S(\log(n)/n)^{1/d}$. Then, we have, for sufficiently large n,

$$\max_{x\in S}\big(\mathbb{E}\hat{f}_n(x)-f(x)\big)\leq \max_{(x,y)\in S^2,\|x-y\|\leq\rho_n}\big(\mathbb{E}\tilde{f}_n(y)-f(x)\big) \text{ a.s.}$$

For all $(x,y)\in S^2$ with $\|x-y\|\leq\rho_n$, we have

$$\mathbb{E}\tilde{f}_n(y)=\int_{\{u:y+uh_n\in S\}}K(u)f(y+uh_n)du.$$

Because f is Lipschitz, we derive that

$$\begin{aligned}\mathbb{E}\tilde{f}_n(y)&\leq\int_{\{u:y+uh_n\in S\}}K(u)\big(f(y)+k_f\|u\|h_n\big)du\\&\leq\int_{\mathbb{R}^d}K(u)\big(f(y)+k_f\|u\|h_n\big)du=f(y)+k_fh_n\int_{\mathbb{R}^d}\|u\|K(u)du.\end{aligned}$$

Now, again using the Lipschitz condition, we have

$$\mathbb{E}\tilde{f}_n(y)\leq f(x)+k_f\rho_n+k_fh_n\int_{\mathbb{R}^d}\|u\|K(u)du.$$

Because $nh_n^d\big(\log(n)\big)^{-2}\big|\log(h_n)\big|^{-1}\to\infty$, we have $h_n\gg\rho_n$. Then, there exists a constant C_2 such that:

$$h_n^{-1}\max_{x\in S}\big(\mathbb{E}\hat{f}(x)-f(x)\big)\leq C_2 \text{ a.s.} \tag{14}$$

The first two conditions on h_n, together with equations (13) and (14), imply that there exists a sequence ε_n such that $\varepsilon_n\log(n)\to 0$ fulfilling

$$\max_{x\in S}\big(\hat{f}_n(x)-f(x)\big)\leq\varepsilon_n.$$

Then, for all $x\in S$, $\hat{f}_n(x)-f(x)\leq f(x)\varepsilon_n/f_0$, and thus, $\frac{\hat{f}_n(x)}{f(x)}\leq 1+\frac{f(x)\varepsilon_n}{f_0}$, or equivalently,

$$\left(\frac{f(x)}{\hat{f}_n(x)}\right)^{1/d}\geq\left(1+\frac{\varepsilon_n}{f_0}\right)^{-1/d}.$$

Finally, if we take $\varepsilon_n^+ = (1-(1+\varepsilon_n/f_0)^{-1/d}) \sim \varepsilon_n/(df_0)$ (thus, we have $\varepsilon_n^+ \log(n) \to 0$) then $\max_{x\in S}\left(\frac{f(x)}{\hat{f}_n(x)}\right)^{1/d} \geq 1-\varepsilon_n^+$ eventually almost surely, which concludes the proof of (i).

We now prove (ii). It is clear that

$$\min_{x\in\mathbb{R}^d} \hat{f}_n(x) \geq \min_{x\in\mathbb{R}^d} \mathbb{E}\hat{f}_n(x) - \max_{x\in\mathbb{R}^d} \left|\mathbb{E}\hat{f}_n(x) - \hat{f}_n(x)\right|.$$

We have already proven $\max_{x\in\mathbb{R}^d}\left|\mathbb{E}\hat{f}_n(x)-\hat{f}_n(x)\right| \to 0$ a.s. using Theorem 2.3 in Giné, E. and Guillou, A. (2002). We now show that $\min_{x\in\mathbb{R}^d}\mathbb{E}\hat{f}_n(x)$ is bounded from below by a positive constant. Observe that $\min_{x\in\mathbb{R}^d}\mathbb{E}\hat{f}_n(x) = \min_{x\in\aleph_n}\mathbb{E}\tilde{f}_n(x)$; thus

$$\min_{x\in\mathbb{R}^d} \mathbb{E}\hat{f}_n(x) \geq \min_{x\in S} \mathbb{E}\tilde{f}_n(x) = \min_{x\in S} \int_{\{u:x+uh_n\in S\}} K(u)f(x+uh_n)du.$$

Using that f is Lipchitz continuous, we obtain:

$$\mathbb{E}\tilde{f}_n(x) \geq \int_{\{u:x+uh_n\in S\}} K(u)\big(f(x)-k_f\|u\|h_n\big)du.$$

Because f is bounded from below and the support is standard with respect to K, we have, for sufficiently large n,

$$\mathbb{E}\tilde{f}_n(x) \geq f_0c_K - k_fh_n\int_{\mathbb{R}^d}\|u\|K(u)du.$$

Therefore, $\min_{x\in\mathbb{R}^d}\hat{f}_n(x) \geq f_0c_K - \varepsilon_n'$ with $\varepsilon_n' \to 0$, thus $\min_{x\in\mathbb{R}^d}\hat{f}_n(x) \geq \lambda - \varepsilon_n^-$ with $\varepsilon_n^- \to 0$ and $\lambda_0 = f_0c_K$.

□

3.3 Simulations

We have performed two simulation studies to asses the behavior of our test in the scenarios described in Sections 3.1 and 3.2. For the first study, the data will be drawn uniformly on sets $S \subset \mathbb{R}^2$, and we will perform the test defined in Section 3.1 to obtain estimations of the power and the level. In the second study, the nonparametric case, the data will be drawn according to an unknown density, and we will estimate the density using the estimator given by (8). In this case, we consider the same sets as in Delicado, P.; Hernández, A. and Lugosi, G. (2014).

3.3.1 Semi-Parametric case

The data are generated uniformly on the sets $S_\varphi = [0,1]^2 \setminus T_\varphi$, where T_φ is the isosceles triangle with height 1/2 (see Figure 1), whose angle at the vertex $(1/2,1/2)$ is equal to φ. If we have a random sample in S_φ, it is clear that as φ increases, it should be easer to detect (with our test 3) the non-convexity of the set. The results of the simulations are summarized in Table 1.

$\varphi = \pi/4$		$\varphi = \pi/6$		$\varphi = \pi/8$	
n	$\hat{\beta}$	n	$\hat{\beta}$	n	$\hat{\beta}$
100	.4	200	.565	300	.543
130	.636	250	.787	350	.679
160	.835	300	.926	400	.846
200	.946	400	.996	500	.976
300	.997	500	1	600	.997

Table 1: Power estimated over 1000 repetitions, for different values of φ, when the sample is uniformly distributed on $[0,1]^2 \setminus T_\varphi$, where T_φ is an isosceles triangle, (see Figure 1)

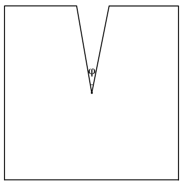

Figure 1: $[0,1]^2 \setminus T_\varphi$ where T_φ is an isosceles triangle with height 1/2.

3.3.2 Non-parametric case

We will perform a simulation study for the same sets used in Delicado, P.; Hernández, A. and Lugosi, G. (2014). Consider the curves $\gamma_{R,\theta} = R(\cos(\theta), \sin(\theta))$ with $\theta \in [\frac{3\pi(R-1)}{2R}, \frac{3}{2}\pi]$ and the reflections of those curves along the y axe (which will be denoted by $\zeta_{R,\theta}$). We consider $\Gamma_R = T_{(0,R)}(\gamma_{R,\theta}) \cup T_{(0,-R)}(\zeta_{R,\theta})$ with $\theta \in [\frac{3\pi(R-1)}{2R}, \frac{3}{2}\pi]$, where T_v is the translation along the vector v. It is easy to see that the length of every Γ_R is $\frac{3}{2}\pi$. We will consider, for different values of R, the S-shaped sets (see first row in Figure 2).

$$S_R = T_{(0,R)}\Big(\bigcup_{R-0.6 \leq r \leq R+0.6} \gamma_{r,\theta}\Big) \cup T_{(0,-R)}\Big(\bigcup_{R-0.6 \leq r \leq R+0.6} \zeta_{r,\theta}\Big)$$

Observe that when R approaches to infinity, the sets S converge to the rectangle (which corresponds to the convex case). We have generated the data according to two different densities. The first one is the same as that considered in Delicado, P.; Hernández, A. and Lugosi, G. (2014): that is, along the orthogonal direction of Γ_R, we choose a random variable with normal density (with zero mean and standard deviation $\sigma = .15$) truncated to .6 (the truncation is performed to ensure that we obtain a point in the set S_R). In the second case, we consider a random variable along the orthogonal direction of Γ_R but uniformly distributed on $[-.6, .6]$. In Tables 2 and 3, we have summarized the results of the simulations, for different sample sizes (we performed the test $B = 100$ times).

R	N=100		N=250		N=500		N=1000	
	np	unif	np	unif	np	unif	np	unif
1	.13	.44	.55	.99	1	1	1	1
1.5	.98	1	1	1	1	1	1	1
3	.38	.24	1	1	1	1	1	1
6	.08	.09	.41	.66	1	1	1	1
12	.01	.05	.02	.08	.39	.68	.98	1
24	0	.07	.01	.05	0	.09	.07	.48
∞	0	.04	0	.09	0	.04	.01	.05

Table 2: Power estimated over B repetitions, for different values of R, when the sample is uniformly distributed along the orthogonal direction of Γ_R

R	N=100		N=250		N=500		N=1000	
	np	unif	np	unif	np	unif	np	unif
1	1	1	1	1	1	1	1	1
1.5	1	1	1	1	1	1	1	1
3	1	.99	1	1	1	1	1	1
6	.67	.41	.99	1	1	1	1	1
12	.25	.19	.62	.98	.85	1	.94	1
24	.1	.30	.30	.92	.38	1	.48	1
∞	0	.33	.04	.92	.06	1	.04	1

Table 3: Power estimated over B repetitions, for different values of R, when the sample is drawn according to a truncated normal distribution (to .6) normal distribution, along the orthogonal direction of Γ_R

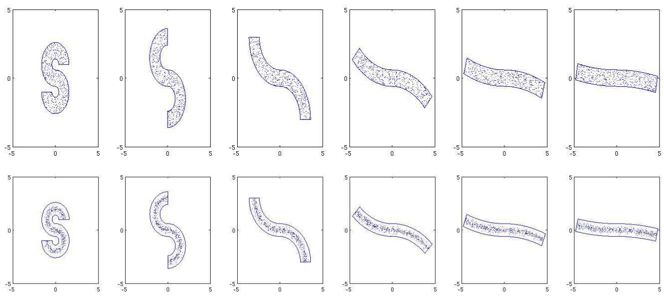

Figure 2: S_R for different values of R. sample is also presented, drawn with a uniform radial noise (top) and with a truncated Gaussian noise (bottom)

4 Appendix

The aim of this Appendix is to prove the main result on the generalization of the maximal spacing, that is, Theorem 2. It is organized as follows: first we settle some preliminary lemmas, then we prove a weaker version of Theorem 2, for the case of piecewise constant densities on disjoint sets. We continue by considering piecewise constant densities, and finally we derive the proof of Theorem 2.

4.1 Preliminary Lemmas

4.1.1 Lemma 1

First we prove, following the ideas in Janson, S. (1987), three technical lemmas from which Corollary 1 is a direct consequence. The first one is useful to control the convergence rates, and it is used all along this section.

Lemma 1. *Let us consider $S \subset \mathbb{R}^d$ with $|S| > 0$, $|\partial S| = 0$, and $\aleph_n = \{X_1, \ldots, X_n\}$ iid random vectors with uniform distribution on S. Then, there exists $a_-^S = a_-^S(w,n)$ and $a_+^S = a_+^S(w,n)$ such that $a_-^S \to \alpha$ and $a_+^S \to \alpha$ if $w \to \infty$ and $w/n \to 0$, and such that,*

$$\exp(-\gamma a_+^S |S|) \leq \mathbb{P}\big(nV(\aleph_n) < w\big) \leq \exp(-\gamma a_-^S |S|), \tag{15}$$

where $\gamma = \frac{n}{|S|} w^{d-1} e^{-w}$.

Observe that the functions a_+^S and a_-^S only depend on the "shape" of S (i.e. are invariant by similarity transformations).

Notation and previous definitions. Let us denote by $\aleph_n = \{X_1, \ldots, X_n\}$ a sample of iid random vectors uniformly distributed on $S \subset \mathbb{R}^d$. We assume that S is compact set. Let $\{N_t\}_{t \geq 0}$ be a Poisson process with intensity 1, independent of $\aleph_n$. Let us denote,

$$\Delta_t = \Delta(\aleph_{N_t}) \quad \text{and} \quad V(t) = \Delta_t^d,$$

where $\Delta(\aleph_{N_t})$ is given in Definition 1. The following characterization is easily derived,

$$\begin{aligned}\Delta(\aleph_n) \geq r \Leftrightarrow \exists x &\text{ such that } x + rA \subset S \setminus \aleph_n \\ \Leftrightarrow \exists x &\text{ such that } x + rA \subset S \textit{ and } x \notin \bigcup_{i=1}^{n} \big(X_i - rA\big).\end{aligned}$$

Therefore, if we define $S_r = \{x : x + rA \subset S\}$, then $\Delta(\aleph_n) < r$ if and only if S_r can be covered by the sets $X_i - rA$. The random set $\{X_i\}_1^{N_t}$ can be considered as a Poisson process with intensity $t/|S|$ in S.
Let us denote by $\mathfrak{F}_s$ the grid $\big\{ \prod_1^d [n_i s, (n_i + 1)s] : (n_1, \ldots, n_d) \in \mathbb{Z}^d \big\}$ and define the following quantities

$$n_s = \#\big\{Q \in \mathfrak{F}_s : Q \subset S_r\big\}, \quad m_s = \#\big\{Q \in \mathfrak{F}_s : Q \cap \partial S_r \neq \emptyset\big\},$$

and

$$\gamma = \gamma\Big(r, \frac{t}{|S|}\Big) = \frac{t^d}{|S|^d} |rA|^{d-1} \exp\Big(-\frac{t|rA|}{|S|}\Big) = \frac{t^d}{|S|^d} r^{d(d-1)} \exp\Big(\frac{-tr^d}{|S|}\Big).$$

Some previous results

Lemma 2. *There exist $a_+ = a_+(tr^d)$ and $a_- = a_-(tr^d)$ such that $a_+ \to \alpha$ and $a_- \to \alpha$ if $tr^d \to \infty$, and for all $s > 3r$,*

$$\exp\Big(-\gamma a_+(s+3r)^d(n_s+m_s)\Big) \leq P(\Delta_t < r) \leq \exp\Big(-\gamma a_-(s-3r)^d n_s\Big). \tag{16}$$

Proof. The proof of (16) follows from Lemma 7.2 in Janson, S. (1986) (let us observe that in Lemma 7.2 it is not used that $|S| = 1$). Replace A and S by rA and S_r respectively. Here $a_- = a_-(-rA, v, t/|S|, 3r)$, $a_+ = a_+(-rA, v, t/|S|, 3r)$ and v is a vector taken conveniently. The fact that $a_+ \to \alpha$ and $a_- \to \alpha$ follows directly from Lemma 7.3 in Janson, S. (1986). □

Lemma 3. *Let $s = \sqrt{r}$. Then, if $r \to 0$, we have $m_s s^d \to 0$ and $n_s s^d \to |S|$.*

Proof. Let us denote $\partial_a S = \partial S \oplus \mathcal{B}(0,a)$. It is easy to see that $\partial S_r \subset \partial_{3r} S$. If $Q \in \mathfrak{F}_s$ and $Q \cap \partial S_r \neq \emptyset$ then $Q \subset \partial_{3r+ds} S$. Thus $m_s s^d \leq |\partial_{3r+ds} S|$. On the other hand $|S| - |\partial_{3r+ds} S| \leq |S \setminus \partial_{3r+ds} S| \leq n_s s^d \leq |S|$. Taking $s = r^{1/2}$, we obtain $|\partial_{3r+ds} S| \to |\partial S| = 0$ if $r \to 0$. Finally $m_s s^d \to 0$ and $n_s s^d \to |S|$. □

As a consequence we obtain the following result.

Lemma 4. *There exist $a_- = a_-(r,t)$ and $a_+ = a_+(r,t)$ fulfilling $a_- \to \alpha$ and $a_+ \to \alpha$ if $r \to 0$ and $tr^d \to \infty$, such that*

$$e^{-\gamma |S| a_+} \leq P(\Delta_t < r) \leq e^{-\gamma |S| a_-}.$$

Now 15) is a direct consequence of the previous Lemma, taking $w = \frac{t}{|S|} r^d$, so that $\Delta_t \leq r \Leftrightarrow \frac{t}{|S|} V(t) \leq w$.

4.1.2 Lemmas 5, 6 and 7

In this section we settle three lemmas whose proofs are quite similar. The first one (Lemma 5), bounds the size of the maximal spacing. It is used in the proofs of Proposition 8 and Theorem 2. Lemmas 6 and 7 controls the speed of the maximal spacing when we constraint the center of the empty set to be localized on a "vanishing" set, under different assumptions on the density. Because A is convex with non-empty interior there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$, $A^{-\varepsilon} \neq \emptyset$ and $|A^{-\varepsilon}| = |A| - \varepsilon |\partial A|_{d-1} + o(\varepsilon)$. It can also be proved easily that,

$$\text{for all } r > 0, \text{ and for all } x \in \mathcal{B}(0, \varepsilon_0/r),\ x + (rA)^{-\|x\|} \subset rA. \tag{17}$$

Lemma 5. *Let $\aleph_n = \{X_1, \dots, X_n\}$ be a random sample of points in $\mathbb{R}^d$, drawn according to a density f with bounded support S. Suppose that f fulfills condition B. Then, there exist a constant r_f such that:*

$$\Delta(\aleph_n) \leq r_f \left(\frac{\log(n)}{n} \right)^{1/d} \quad \textit{eventually almost surely.}$$

Proof. Let us first cover S with $\nu_n \le C_S n^{-1}$ balls of radii $n^{-1/d}$ centered at $\{x_1, \dots, x_{\nu_n}\}$, and let $w_n = \left(\frac{r_f \log(n)}{n}\right)^{1/d}$ with $r_f > 2f_1/f_0$. We are going to prove that $\Delta(\aleph_n) \le w_n$, eventually almost surely. Because $\Delta(\aleph_n) \ge w_n \Leftrightarrow \exists x \in S,$ such that $x + w_n f(x)^{-1/d} A \subset S \setminus \aleph_n$, then $\Delta(\aleph_n) \ge w_n \Rightarrow \exists x \in S,$ such that $x + w_n f_1^{-1/d} A \subset S \setminus \aleph_n$. There exists a point x_i such that $\|x - x_i\| \le n^{-1/d}$ and, for sufficiently large n, by (17) (because $n^{-1/d} \ll w_n$) we have that $x_i + (w_n f_1^{-1/d} A)^{-1/n^{1/d}} \subset S \setminus \aleph_n$. Thus,

$$\Delta(\aleph_n) \ge w_n \Rightarrow \exists x_i \; \exists x \in \mathcal{B}(x_i, n^{-1/d}), \text{ such that } x_i + (w_n f_1^{-1/d} A)^{-1/n^{1/d}} \subset S \setminus \aleph_n. \quad (18)$$

Now notice that,

$$\begin{aligned} \mathbb{P}\Big(x_i + \big(w_n f_1^{-1/d} A\big)^{-1/n^{1/d}} \subset S \setminus \aleph_n\Big) &= \left(1 - \mathbb{P}\Big(x_i + \big(w_n f_1^{-1/d} A\big)^{-1/n^{1/d}}\Big)\right)^n \\ &\le \left(1 - f_0 \Big|\big(w_n f_1^{-1/d} A\big)^{-1/n^{1/d}}\Big|\right)^n \\ &\le \left(1 - \Big(\frac{f_0}{f_1} w_n^d - \frac{f_0}{f_1^{\frac{d-1}{d}}} w_n^{d-1} n^{-1/d} (1 + o(1))\Big)\right)^n. \end{aligned}$$

In the last inequality we used that $|A^{-\varepsilon}| = |A| - \varepsilon |\partial A|_{d-1} + o(\varepsilon)$. Because $n^{-1/d} \ll w_n$, we have that,

$$\mathbb{P}\Big(x_i + \big(w_n f_1^{-1/d} A\big)^{-1/n^{1/d}} \subset S \setminus \aleph_n\Big) \le \left(1 - \frac{f_0}{f_1} w_n^d (1 + o(1))\right)^n.$$

From this inequality and (18) we obtain that,

$$\begin{aligned} \mathbb{P}\Big(\Delta(\aleph_n) \ge r_f \big(\log(n) n/n\big)^{1/d}\Big) &\le \nu_n \Big(1 - \frac{f_0}{f_1} w_n^d (1 + o(1))\Big)^n \\ &\le \nu_n \exp\big(-c n w_n^d (1 + o(1))\big), \end{aligned}$$

and therefore,

$$\mathbb{P}\Big(\Delta(\aleph_n) \ge r_f (\log(n)/n)^{1/d}\Big) \le C_S n^{1 - r_f f_0/f_1 + o(1)}.$$

Finally, because $r_f > 2f_1/f_0$ we have $\sum \mathbb{P}\big(\Delta(\aleph_n) \ge r_f (\log(n)/n)^{1/d}\big) < \infty$. Thus, the Borel-Cantelli Lemma ensures that $\Delta(\aleph_n) \le r_f (\log(n)/n)^{1/d}$ eventually almost surely □

Lemma 6. *Let $\aleph_n = \{X_1, \dots, X_n\}$ be a random sample of points in $\mathbb{R}^d$, drawn according to a density f, which is assumed to fulfill condition (B). Suppose also that there exist constants r_0 and $c > 1 - 1/d$ such that for all $r \le r_0$ and for all $x \in S$:*

$$\frac{\min_{t \in S \cap \mathcal{B}(x,r)} f(t)}{\max_{t \in S \cap \mathcal{B}(x,r)} f(t)} \ge c.$$

Let G_n be a sequence of sets with the following property: the number of balls of radius $n^{-1/d}$, necessary to cover G_n (which we will denote ν_n), satisfies $\nu_n \leq n^{1-d^{-1}}(\log(n))^\beta$ for some β. Let A be a compact and convex set with $|A| = 1$ such that its barycenter is the origin of $\mathbb{R}^d$. Let us denote

$$\begin{aligned}
\Delta(\aleph_n, G_n) &= \sup\Big\{r : \ \exists x \in G_n \text{ such that } x + \frac{r}{f(x)^{1/d}} A \subset S \setminus \aleph_n\Big\},\\
V(\aleph_n, G_n) &= \Delta^d(\aleph_n, G_n),\\
U(\aleph_n, G_n) &= nV(\aleph_n, G_n) - \log(n) - (d-1)\log(\log(n)) - \log(\alpha).
\end{aligned}$$

Then, for all $x \in \mathbb{R}$ we have that

$$\mathbb{P}\big(U(\aleph_n, G_n) \geq x\big) \to 0.$$

Proof. Let us first cover G_n with ν_n balls of radius $n^{-1/d}$, centered at some points $\{x_1, \dots, x_{\nu_n}\}$, and choose

$$w_n = \Big(\frac{x + \log(n) + (d-1)\log(\log(n)) + \log(\alpha)}{n}\Big)^{1/d},$$

then $\Delta(\aleph_n) \geq w_n \Leftrightarrow U(\aleph_n) \geq x$. Then, it also holds that

$$\Delta(\aleph_n) \geq w_n \Leftrightarrow \exists x \in G_n, \text{ such that } x + w_n f(x)^{-1/d} A \subset S \setminus \aleph_n.$$

On the other hand, because $n^{-1/d} \ll w_n$ and $f(x)^{-1/d} \leq f_0^{-1/d}$, there exists a point x_i such that $\|x - x_i\| \leq n^{-1/d}$ and, for sufficiently large n, $x_i + (w_n f(x)^{-1/d} A)^{-1/n^{1/d}} \subset S \setminus \aleph_n$. And then,

$$\Delta(\aleph_n) \geq w_n \Rightarrow \exists x_i\ \exists x \in \mathcal{B}(x_i, n^{-1/d}), \text{ such that} \\ x_i + (w_n f(x)^{-1/d} A)^{-1/n^{1/d}} \subset S \setminus \aleph_n. \quad (19)$$

Now observe that

$$\begin{aligned}
\mathbb{P}\left(x_i + (w_n f(x)^{-1/d} A)^{-1/n^{1/d}} \subset S \setminus \aleph_n\right) &= \left(1 - \mathbb{P}\Big(x_i + \big(w_n f(x)^{-1/d} A\big)^{-1/n^{1/d}}\Big)\right)^n\\
&\leq \left(1 - \frac{\min_{t \in x_i + w_n f_0^{-1/d} A} f(t)}{\max_{t \in \mathcal{B}(x_i, n^{-1/d})} f(t)} w_n^d (1 + o(1))\right)^n,
\end{aligned}$$

which implies that, for sufficiently large n,

$$\mathbb{P}\left(x_i + (w_n f(x)^{-1/d} A)^{-1/n^{1/d}} \subset S \setminus \aleph_n\right) \leq \Big(1 - c w_n^d (1 + o(1))\Big)^n. \quad (20)$$

From this inequality, together with (19) we derive that,

$$\begin{aligned}
\mathbb{P}\big(U(\aleph_n, G_n) \geq x\big) &\leq \nu_n (1 - c w_n^d (1 + o(1)))^n\\
&\leq \nu_n \exp\big(-c n w_n^d (1 + o(1))\big)\\
&\leq \nu_n n^{-c(1+o(1))}.
\end{aligned}$$

Finally, $\mathbb{P}(U(\aleph_n, G_n) \geq x) \to 0$ because $c > 1 - 1/d$ and $\nu_n \leq n^{1-1/d}(\log(n))^a$. □

Lemma 7. *Let $\aleph_n = \{X_1, \dots, X_n\}$ be a random sample of points in $\mathbb{R}^d$, drawn according to a density f, and assume that condition B holds. Let G_n be a sequence of sets with the following property: the number of balls of radius $n^{-1/d}$, necessary to cover G_n (which we will denote ν_n), satisfies $\nu_n \leq n^{1-a}$ for some $a > 0$. Let A be a compact and convex set with $|A| = 1$, whose barycentre is the origin of $\mathbb{R}^d$. If we denote*

$$\begin{aligned}
\Delta(\aleph_n, G_n) &= \sup\Big\{r : \ \exists x \in G_n \text{ such that } x + \frac{r}{f(x)^{1/d}} A \subset S \setminus \aleph_n\Big\}, \\
V(\aleph_n, G_n) &= \Delta^d(\aleph_n, G_n) \\
U(\aleph_n, G_n) &= nV(\aleph_n, G_n) - \log(n) - (d-1)\log(\log(n)) - \log(\alpha),
\end{aligned}$$

then, for all $x \in \mathbb{R}$:

$$\mathbb{P}\big(U(\aleph_n, G_n) \geq x\big) \to 0.$$

Proof. The proof is similar to the proof of Lemma 6. Equation (19) also holds, but now(20) becomes:

$$\mathbb{P}\left(x_i + (w_n f(x)^{-1/d} A)^{-1/n^{1/d}} \subset S \setminus \aleph_n\right) \leq \Big(1 - w_n^d(1 + o(1))\Big)^n.$$

So $\mathbb{P}\big(U(\aleph_n, G_n) \geq x\big) \leq \nu_n n^{-1+o(1)}$. Because $\nu_n \leq n^{1-a}$, we have $\mathbb{P}\big(U(\aleph_n, G_n) \geq x\big) \to 0$ □

4.1.3 Lemma 8

This last preliminary relates the behavior of the maximal spacing for a bounded density with the maximal spacing of the uniform density, and it is only used in the proof of the last theorem.

Lemma 8. *Let us consider a density f with compact support S such that, for all $x \in S$ $1 - \varepsilon \leq f(x)|S| \leq 1 + \varepsilon$ for a given $\varepsilon \in (0, 1/2)$. Denote by $n_0 = \lfloor n(1 - 2\varepsilon) \rfloor$ and $n_1 = \lceil n(1 + 2\varepsilon) \rceil$ the floor and ceiling of $n(1 - 2\varepsilon)$ and $n(1 + 2\varepsilon)$ respectively. Fixed $w \in \mathbb{R}$, let $w_0 = \frac{w(1-2\varepsilon-n^{-1})}{(1+\varepsilon)}$ and $w_1 = \frac{w(1-\varepsilon)}{1+2\varepsilon}$ then,*

$$\mathbb{P}\big(n_0 V(\mathcal{Y}_{n_0}) \leq w_0\big)\Big(1 - \frac{1-\varepsilon}{n\varepsilon}\Big) \leq \mathbb{P}\big(nV(\aleph_n) \leq w\big) \tag{21}$$

and

$$\mathbb{P}\big(nV(\aleph_n) \leq w\big) \leq \mathbb{P}\big(n_1 V(\mathcal{Y}_{n_1}) \leq w_1\big)\Big(1 - \frac{1 + 2\varepsilon + n^{-1}}{(n\varepsilon + 1)(1 + \varepsilon)}\Big)^{-1}, \tag{22}$$

where $\mathcal{Y}_{n_1} = \{Y_1, \dots, Y_{n_1}\}$ and $\mathcal{Y}_{n_0} = \{Y_1, \dots, Y_{n_0}\}$ are iid random vectors with uniform distribution on S and $\aleph_n = \{X_1, \dots, X_n\}$ are iid random vectors with density f.

Proof. We first prove (21). To do so, observe that X can be generated from the following mixture: with probability $p = 1 - \varepsilon$, X is drawn with uniform distribution on S, and, with probability $1 - p = \varepsilon$, X is drawn with the law given by the density $g(x) =$

$\frac{f(x)|S|-(1-\varepsilon)}{\varepsilon|S|}\mathbb{I}_S(x)$. Let us denote N_0 the number of points drawn according to the uniform law on S and $\aleph^*_{N_0}=\{Y_1,\dots,Y_{N_0}\}$ the associated sample. Let us recall that

$$\Delta(\aleph_n)=\sup\Big\{r:\ \exists x \text{ such that } x+\frac{r}{f(x)^{1/d}}A\subset S\setminus\aleph_n\Big\}.$$

Because $f(x)|S|\leq 1+\varepsilon$, if we multiply and divide by $|S|^{1/d}$ we have:

$$\Delta(\aleph_n)\leq(1+\varepsilon)^{1/d}\sup\Big\{r:\ \exists x \text{ such that } x+\frac{r}{|S|^{1/d}}A\subset S\setminus\aleph_n\Big\}.$$

Then, $\aleph_{N_0}\subset\aleph_n$ implies that,

$$\Delta(\aleph_n)\leq(1+\varepsilon)^{1/d}\sup\Big\{r:\ \exists x \text{ such that } x+\frac{r}{|S|^{1/d}}A\subset S\setminus\aleph^*_{N_0}\Big\},$$

and therefore $\Delta(\aleph_n)\leq(1+\varepsilon)^{1/d}\Delta(\aleph^*_{N_0})$, which entails that $V(\aleph_n)\leq(1+\varepsilon)V(\aleph^*_{N_0})$. Then, for all w,

$$\mathbb{P}\big(nV(\aleph_n)\leq w\big)\geq\mathbb{P}\big((1+\varepsilon)nV(\aleph^*_{N_0})\leq w\big),$$

and in particular for any n_0 we have that

$$\mathbb{P}\big(nV(\aleph_n)\leq w\big)\geq\mathbb{P}\Big(\big((1+\varepsilon)nV(\aleph^*_{N_0})\leq w\big)\cap\big(N_0\geq n_0\big)\Big).$$

When $N_0\geq n_0$, let us denote $\mathcal{Y}_{n_0}=\{Y_1,\dots,Y_{n_0}\}$ the n_0 first values of $\aleph^*_{N_0}$. Clearly we have $V(\aleph^*_{N_0})\leq V(\mathcal{Y}_{n_0})$ so,

$$\mathbb{P}^{N_0\geq n_0}\left((1+\varepsilon)nV(\aleph^*_{N_0})\leq w\right)\geq\mathbb{P}\Big((1+\varepsilon)nV(\mathcal{Y}_{n_0})\leq w\Big),$$

where $\mathbb{P}^{N_0\geq n_0}$ denotes the probability conditioned to $N_0\geq n_0$. Therefore,

$$\mathbb{P}\big(nV(\aleph_n)\leq w\big)\geq\mathbb{P}\Big(n_0V(\mathcal{Y}_{n_0})\leq\frac{wn_0}{(1+\varepsilon)n}\Big)\mathbb{P}(N_0\geq n_0).$$

On the other hand, because $N_0\sim Bin((1-\varepsilon,n))$, we obtain,

$$\mathbb{P}(N_0<n_0)=\mathbb{P}\Big(N_0-(1-\varepsilon)n<n_0-(1-\varepsilon)n\Big).$$

From $n_0\leq n(1-\varepsilon)$ it follows that $n_0-n(1-\varepsilon)\leq-\varepsilon n$, and by Tchebichev inequality,

$$\mathbb{P}(N_0<n_0)\leq\frac{n\varepsilon(1-\varepsilon)}{n^2\varepsilon^2}=\frac{(1-\varepsilon)}{n\varepsilon},$$

and

$$\mathbb{P}(N_0\geq n_0)\geq 1-\frac{(1-\varepsilon)}{n\varepsilon}.$$

Let us denote $w_0=\frac{w(1-2\varepsilon-n^{-1})}{(1+\varepsilon)}$. Because $n(1-2\varepsilon)-1\leq n_0$ we have $w_0\leq\frac{wn_0}{(1+\varepsilon)n}$, from where it follows that

$$\mathbb{P}\big(nV(\aleph_n)\leq w\big)\geq\mathbb{P}\Big(n_0V(\mathcal{Y}_{n_0})\leq w_0\Big)\left(1-\frac{1-\varepsilon}{n\varepsilon}\right).$$

Equation (22) is proved in the same way as (21). We provide a sketch of the proof. First observe that a variable Y with uniform distribution can be seen as a mixture. We take with probability $p = \frac{1}{1+\varepsilon}$, Y as a random variable with the law given by a density f, and, with probability $1-p = \frac{\varepsilon}{1+\varepsilon}$, Y is drawn with the law given by $g(x) = \frac{1+\varepsilon-|S|f(x)}{\varepsilon S}\mathbb{I}_S(x)$. Then, following the ideas used in the proof of equation (21) we consider a sample $\mathcal{Y}_{n_1} = \{Y_1,\dots,Y_{n_1}\}$ of iid copies of Y, (that follows an uniform law). Denote by N the number of the points that had been drawn according to the density f and $\mathcal{Y}^*_N = \{X_1,\dots X_N\}$ these points. The rest of the proof follow the same argument used to prove ((21). $\square$

4.2 Uniform mixture on disjoint supports

Proposition 7. *Let $E_1,\dots,E_k$ be disjoint subsets of $\mathbb{R}^d$, (i.e: $i \neq j \Rightarrow \overline{E_i}\cap\overline{E_j} = \emptyset$), whose Lebesgue measure satisfies $0 < |E_i| < \infty$. Let $p_1,\dots,p_k$ be positive real numbers. If $\aleph_n = \{X_1,\dots,X_n\}$ is a random sample of points in $\mathbb{R}^d$, drawn according to the density:*

$$f(x) = \sum_{i=1}^{k} p_i \mathbb{I}_{E_i}(x),$$

then:

$$U(\aleph_n) \xrightarrow{\mathcal{L}} U \quad \text{when } n\to\infty \qquad \text{where, for all } i\ 0 < p_i < \infty.$$

Proof. First let us introduce some notation:

- $N_i = \#\{\aleph_n \cap E_i\}$, the number of points in E_i, which has a Binomial distribution, $N_i \sim Bin(n, p_i|E_i|)$.
- $\aleph^i_{N_i} = \{X_{i_1},\dots,X_{i_{N_i}}\}$, the subsample of $\aleph_n$ that belongs to E_i. Observe that X_{i_j} for $j = 1,\dots,N_i$ has uniform distribution on E_i with density $|E_i|^{-1}$.
- $a_i = p_i|E_i|$, that fulfills $\sum a_i = 1$, $a_0 = \min_i a_i$, $A_0 = \max_i a_i$ and $C = \sum \frac{1-a_i}{a_i}$.
- $\varepsilon_i = \frac{N_i - a_i n}{n a_i}$.

Because the support S of f is equal to $\cup_i \overline{E_i}$, by assumption $i \neq j \Rightarrow \overline{E_i}\cap\overline{E_j} = \emptyset$, we have

$$\Delta(\aleph_n) = \sup\Big\{r:\ \exists x \exists i \text{ such that } x + \frac{r}{p_i^{1/d}} A \subset E_i \setminus \aleph_n\Big\},$$

so

$$\Delta(\aleph_n) = \max_i \sup\Big\{r:\ \exists x \text{ such that } x + \frac{r|E_i|^{1/d}}{(|E_i|p_i)^{1/d}} A \subset E_i \setminus \aleph^i_{N_i}\Big\}, \tag{23}$$

while

$$\Delta(\aleph^i_{N_i}) = \sup\Big\{r':\ \exists x \in E_i \text{ such that } x + r'|E_i|^{1/d} A \subset E_i \setminus \aleph^i_{N_i}\Big\}. \tag{24}$$

From (23) and (24) we derive that

$$\Delta(\aleph_n) = \max_i \Big\{(|E_i|p_i)^{1/d}\Delta(\aleph^i_{N_i})\Big\},$$

and

$$V(\aleph_n) = \max_i \Big\{(|E_i|p_i)V(\aleph^i_{N_i})\Big\},$$

which entails that

$$\mathbb{P}(V(\aleph_n) \le w) = \prod_i \mathbb{P}\Big(N_i V(\aleph^i_{N_i}) \le \frac{wN_i}{a_i n}\Big). \tag{25}$$

Let us now condition to the number of points in each E_i. Denoting $\mathbb{P}^{\vec{n}}(A) = \mathbb{P}(A|N_1 = n_1, \dots, N_k = n_k)$, we have that,

$$\mathbb{P}^{\vec{n}}\big(nV(\aleph_n) \le w\big) = \prod_{i=1}^k \mathbb{P}^{\vec{n}}\big(n(|E_i|p_i)V(\aleph^i_{n_i}) \le w\big) = \prod_{i=1}^k \mathbb{P}^{\vec{n}}\left(n_i V(\aleph^i_{n_i}) \le \frac{wn_i}{n|E_i|p_i}\right).$$

Now, taking $w_i = \frac{wn_i}{n|E_i|p_i}$, $\gamma_i = \frac{n_i w_i^{d-1}e^{-w_i}}{|E_i|}$ and applying Lemma 1 we obtain,

$$\exp\bigg(-\sum_{i=1}^k \gamma_i a_+^{E_i}|E_i|\bigg) \le \mathbb{P}^{\vec{n}}\big(nV(\aleph_n) \le w\big) \le \exp\bigg(-\sum_{i=1}^k \gamma_i a_-^{E_i}|E_i|\bigg).$$

On the other hand,

$$\begin{aligned}
\sum_{i=1}^k \gamma_i a_+^{E_i}|E_i| &= \sum_{i=1}^k n_i \left(\frac{wn_i}{n|E_i|p_i}\right)^{d-1} \exp\left(-\frac{wn_i}{n|E_i|p_i}\right) a_-^{E_i} \\
&= \sum_{i=1}^k n_i w^{d-1}(1+\varepsilon_i)^{d-1}\exp(-w(1+\varepsilon_i))a_+^{E_i}(w_i, n_i)
\end{aligned}$$

Let $\varepsilon = \max_i |\varepsilon_i|$ and $\varepsilon_{a_+} = \max_i \frac{|a_+^{E_i}(w_i,n_i)-\alpha|}{\alpha}$, then we have

$$\sum_{i=1}^k \gamma_i a_+^{E_i}|E_i| \le nw^{d-1}\exp(-w)\alpha(1+\varepsilon)^{d-1}\exp(w\varepsilon)(1+\varepsilon_{a_+}). \tag{26}$$

Taking $w = x + \log(n) + (d-1)\log(\log(n)) + \log(\alpha)$, we obtain that $nV \le w \Leftrightarrow U \le x$, which implies that

$$\mathbb{P}^{\vec{n}}\big(U(\aleph_n) \le x\big) \ge \exp\Big(-e^{-x}(1+\varepsilon)^{d-1}\exp(\log(n)\varepsilon)(1+\varepsilon_{a_+})(1+o_n(1))\Big). \tag{27}$$

In the same way it can be proved that (denoting $\varepsilon_{a_-} = \max_i \frac{|a_-^{E_i}(w_i,n_i)-\alpha|}{\alpha}$):

$$\mathbb{P}^{\vec{n}}\big(U(\aleph_n) \le x\big) \le \exp\Big(-e^{-x}(1-\varepsilon)^{d-1}\exp(-\log(n)\varepsilon)(1-\varepsilon_{a_-})(1+o_n(1))\Big).$$

If $\varepsilon = \max|\varepsilon_i| \leq 1/\log(n)^2$, then, if $n \geq 5$, $a_0 n/2 \leq N_i \leq n$ for all i. So $w_i \geq \log(n)a_0/(2A_0) \to \infty$ and $w_i/n \leq \big(x+\log(n)+(d-1)\log(\log(n))+\log(\alpha)\big)/(na_0) \to 0$. This implies that ε_{a_-} and ε_{a_+} converges to 0, according to Lemma 8. Then we have

$$\mathbb{P}^{\varepsilon \leq 1/\log(n)^2}\big(U(\aleph_n) \leq x\big) \to \exp(-\exp(-x)) \quad \text{when } n \to \infty. \tag{28}$$

Because

$$\mathbb{P}\Big(\max_i |\varepsilon_i| \geq \frac{1}{\log(n)^2}\Big) = \mathbb{P}\Big(\bigcup_i |\varepsilon_i| \geq \frac{1}{\log(n)^2}\Big) \leq \sum_{i=1}^{k} \mathbb{P}\Big(|\varepsilon_i| \geq \frac{1}{\log(n)^2}\Big), \tag{29}$$

from Tchebychev inequality we obtain

$$\mathbb{P}\Big(|\varepsilon_i| \geq \frac{1}{\log(n)^2}\Big) \leq \log(n)^4 \mathbb{V}(\varepsilon_i^2) \quad \text{where} \quad \mathbb{V}(\varepsilon_i^2) = \frac{1-a_i}{na_i},$$

and therefore

$$\mathbb{P}\Big(\varepsilon \geq \frac{1}{\log(n)^2}\Big) \leq C\frac{\big(\log(n)\big)^4}{n}. \tag{30}$$

Finally, from equations (28) and (30) we obtain that

$$\mathbb{P}\big(U(\aleph_n) \leq x\big) \to \exp(-\exp(-x)) \text{ when } n \to \infty,$$

which concludes the proof. □

4.3 Uniform mixture

Proposition 8. *Let $E_1, \ldots, E_k$ be subsets of $\mathbb{R}^d$ such that:*

1) $i \neq j \Rightarrow |\overline{E_i} \cap \overline{E_j}| = 0$.
2) $0 < |E_i| < \infty$ *for* $i = 1, \ldots, k$.
3) *There exists* $0 < K < \infty$ *such that,* $|\partial E_i|_{d-1} \leq K$ *for* $i = 1, \ldots, k$*, where* $|\partial E_i|_{d-1}$ *is the* $d-1$ *measure of the boundary of* E_i.

Suppose that $\aleph_n = \{X_1, \ldots, X_n\}$ is a random sample of points in $\mathbb{R}^d$, drawn according to the density:

$$f(x) = \sum_{i=1}^{k} p_i \mathbb{I}_{\mathring{E}_i}$$

where $p_1, \ldots, p_k$ are real numbers satisfying $0 < p_i < \infty$ for $i = 1, \ldots, k$. If there exists constants $r_0 > 0$ and $c > 1 - 1/d$ such that, for all $r \leq r_0$ and all $x \in \mathring{E}_i$ for some i,

$$\frac{\min_{t \in S \cap B(x,r)} f(t)}{\max_{t \in S \cap B(x,r)} f(t)} \geq c.$$

then:

$$U(\aleph_n) \xrightarrow{\mathcal{L}} U \quad \textit{whenn} \to \infty.$$

Proof. We start by introducing some definitions and notation. Let

$$\begin{aligned}
\mathring{\Delta}(\aleph_n) &= \sup\Big\{r:\ \exists x \exists i,\ \text{ such that } x + \frac{r}{f(x)^{1/d}}A \subset \mathring{E}_i \setminus \aleph_n\Big\},\\
\mathring{V}(\aleph_n) &= \mathring{\Delta}^d(\aleph_n),\\
\mathring{U}(\aleph_n) &= n\mathring{V}(\aleph_n) - \log(n) - (d-1)\log\big(\log(n)\big) - \log(\alpha_A)
\end{aligned}$$

Clearly $U(\aleph_n) \geq \mathring{U}\big(\aleph_n\big)$, and therefore $\mathbb{P}\big(U(\aleph_n) \leq x\big) \leq \mathbb{P}\big(\mathring{U}\big(\aleph_n\big) \leq x\big)$.

It can be proved, following the same ideas used to prove Theorem 7 (and the fact that $|E_i| = |\mathring{E}_i|$) that $\mathring{U}\big(\aleph_n\big) \xrightarrow{\mathcal{L}} U$. We denote by $F_n(x) = \mathbb{P}\big(\mathring{U}\big(\aleph_n\big) \leq x\big)$ and by $G = \bigcup_{i,j}\big(\overline{E_i} \cap \overline{E_j}\big)$, and define the following quantities:

- $p_0 = \min_i p_i$.
- $\rho_A = \max_{x\in A}\|x\|$.
- $\rho_n = (r_f\rho_A/p_0^{1/d})\big(\log(n)/n\big)^{1/d}$.
- $\Delta\big(\aleph_n, S\setminus G^{\rho_n}\big) = \sup\Big\{r:\ \exists x \in S\setminus G^{\rho_n} \text{ such that } x + \frac{r}{f(x)^{1/d}}A \subset S\setminus\aleph_n\Big\}$.
- $\Delta\big(\aleph_n, G^{\rho_n}\big) = \sup\Big\{r:\ \exists x \in G^{\rho_n} \text{ such that } x + \frac{r}{f(x)^{1/d}}A \subset S\setminus\aleph_n\Big\}$.

It follows easily that

$$\Delta(\aleph_n) = \max\Big\{\Delta\big(\aleph_n, S\setminus G^{\rho_n}\big), \Delta\big(\aleph_n, G^{\rho_n}\big)\Big\}. \tag{31}$$

According to Lemma 5 there exists a constant r_f such that $\Delta(\aleph_n) \leq r_f\big(\log(n)/n\big)^{1/d}$ eventually almost surely. For the chosen ρ_n, we claim that

$$\Delta\big(\aleph_n, S\setminus G^{\rho_n}\big) \leq \mathring{\Delta}\big(\aleph_n\big) \text{ eventually almost surely.} \tag{32}$$

In order to prove (32) let us observe first that

$$\text{for all } \varepsilon > 0 \text{ there exists } x_\varepsilon \in S\setminus G^{\rho_n} \text{ such that } x_\varepsilon + \frac{\Delta\big(\aleph_n, S\setminus G^{\rho_n}\big) - \varepsilon}{f(x_\varepsilon)^{1/d}}A \subset S\setminus\aleph_n.$$

If $\Delta\big(\aleph_n, S\setminus G^{\rho_n}\big) \leq r_f(\log(n)/n)^{1/d}$ then

$$x_\varepsilon + \frac{\Delta\big(\aleph_n, S\setminus G^{\rho_n}\big) - \varepsilon}{f(x_\varepsilon)^{1/d}}A \subset \mathcal{B}\left(x_\varepsilon, \rho_A\frac{r_f(\log(n)/n)^{1/d} - \varepsilon}{p_0^{1/d}}\right).$$

Because $d(x_\varepsilon, G) \geq r_f\rho_A p_0^{-1/d}(\log(n)/n)^{1/d}$ we have

$$x_\varepsilon + \frac{\Delta\big(\aleph_n, S\setminus G^{\rho_n}\big) - \varepsilon}{f(x_\varepsilon)^{1/d}}A \subset \bigcup_i \mathring{E}_i \setminus \aleph_n.$$

Then, if $\Delta\big(\aleph_n, S\backslash G^{\rho_n}\big) \leq r_f(\log(n)/n)^{1/d}$ it follows that for all $\varepsilon > 0$, $\mathring{\Delta}\big(\aleph_n\big) \geq \Delta\big(\aleph_n, S\backslash G^{\rho_n}\big) - \varepsilon$, and finally, (32) is a direct consequence from the fact that $\Delta\big(\aleph_n, S \setminus G^{\rho_n}\big) \leq \Delta(\aleph_n) \leq r_f(\log(n)/n)^{1/d}$ eventually almost surely.
Using now (31) we obtain

$$\begin{aligned}
&\mathbb{P}(U(\aleph_n) \geq x) \leq \\
&\quad \mathbb{P}^{\Delta(\aleph_n, S\backslash G^{\rho_n}) \leq \mathring{\Delta}(\aleph_n)}\Big(\max\big\{\mathring{U}(\aleph_n), U(\aleph_n, G^{\rho_n})\big\} \geq x\Big)\mathbb{P}\Big(\Delta(\aleph_n, S \setminus G^{\rho_n}) \leq \mathring{\Delta}(\aleph_n)\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \mathbb{P}\Big(\Delta(\aleph_n, S \setminus G^{\rho_n}) \geq \mathring{\Delta}(\aleph_n)\Big),
\end{aligned}$$

which entails

$$\mathbb{P}\big(U(\aleph_n) \geq x\big) \leq 1 - F_n(x) + \mathbb{P}\big(U(\aleph_n, G^{\rho_n}) \geq x\big) + \mathbb{P}\big(\Delta(\aleph_n, S \setminus G^{\rho_n}) \geq \mathring{\Delta}(\aleph_n)\big).$$

Because $\mathbb{P}\big(\Delta(\aleph_n, S \setminus G^{\rho_n}\big) \geq \mathring{\Delta}(\aleph_n)) \to 0$ and $F_n(x) \to \exp(-\exp(-x))$, it only remains to prove that $\mathbb{P}\big(U(\aleph_n, G^{\rho_n}) \geq x\big) \to 0$. In order to do that, we will see that G^{ρ_n} can be covered by a suitable number of balls of radius $1/n^{1/d}$, and then we will apply Lemma 6.

Because $|\partial E_i| < K$ for $i = 1, \dots, k$, every ∂E_i can be covered by $\nu_1 \leq K\rho_n^{-d+1}$ balls of radius ρ_n centered at some points x_i. Every ball $\mathcal{B}(x_i, \rho_n)$ can be covered by $\nu_2 \leq c^*\rho_n^d n$ balls of radius $(1/n)^{1/d}$. Finally, because $G^{\rho_n} \subset \bigcup_i (\partial E_i)^{\rho_n}$, the set G^{ρ_n} can be covered by less than $kKc^*\rho_n n = \mathcal{O}(n^{1-1/d}(\log(n))^{1/d})$ balls of radius $1/n$. That conclude the proof. □

4.4 Lipshitz continous density

Now we will prove a generalization of the Theorem 1 to the case of Lipschitz densities with compact support. Let us recall here the theorem:

Theorem. *Let f be a density with compact support $S \subset \mathbb{R}^d$, let us assume that condition B holds, then*

$$U(\aleph_n) \xrightarrow{\mathcal{L}} U \quad \textit{when } n \to \infty. \tag{33}$$

$$\liminf_{n\to+\infty} \frac{nV(\aleph_n) - \log(n)}{\log(\log(n))} = d - 1 \ \textit{a.s.} \tag{34}$$

$$\limsup_{n\to+\infty} \frac{nV(\aleph_n) - \log(n)}{\log(\log(n))} = d + 1 \ \textit{a.s.} \tag{35}$$

Proof. We will only prove (33), the proof of (34) and (35) are the same as the one in Janson, S. (1987). What we will do is to combine all the methods used to prove the previous theorems. Let us consider a "mesh" of $\mathbb{R}^d$ with small squares of side c_n,

$$\prod_{i=1}^{d} \Big[k_i c_n\ , (k_i + 1)c_n\Big] \quad \text{with } k_i \in \mathbb{N}.$$

We will suppose that $c_n = \mathcal{O}\big((\log(n)/n)^{\frac{1}{3d}}\big)$. Let us denote m_n the number of this squares that are included in S and $C_1, \dots, C_{m_n}$ this squares.

First inequality Like in the proof of Proposition 8 let us denote,

$$\begin{aligned}
\mathring{\Delta}(\aleph_n) &= \sup\Big\{r: \ \exists x \exists i, \ \text{ such that } x + \frac{r}{f(x)^{1/d}} A \subset \mathring{C}_i \setminus \aleph_n\Big\},\\
\mathring{V}(\aleph_n) &= \mathring{\Delta}^d(\aleph_n),\\
\mathring{U}(\aleph_n) &= n\mathring{V}(\aleph_n) - \log(n) - (d-1)\log\big(\log(n)\big) - \log(\alpha_A).
\end{aligned}$$

Because $\bigcup_{i=1}^{m_n} \mathring{C}_i \subset S$ we have: $\mathbb{P}\big(U(\aleph_n) \le x\big) \le \mathbb{P}\Big(\mathring{U}(\aleph_n) \le x\Big)$.
Also, like in the proof of Proposition 7 let us denote:

- $N_i = \#\{\aleph_n \cap C_i\}$.
- $a_i = \int_{C_i} f(t)dt$; $a_0 = \min_i a_i$; $A_0 = \max_i a_i$ and $C = \sum \frac{1-a_i}{a_i}$. Observe that $\sum a_i = 1$ and $a_0 \ge f_0 c_n^d$.
- $\aleph_{N_i}^i = \{X_{i_1}, \dots, X_{i_{N_i}}\}$ the subsample of $\aleph_n$ that belongs to C_i. Observe that X_{i_j} for $j = 1, \dots, N_i$ has density $f_i(x) = \big(f(x)/a_i\big)\mathbb{I}_{C_i}(x)$.
- $\varepsilon_i = \frac{N_i - a_i n}{n a_i}$

Proceeding exactly as in the proof of Proposition 7 we can derive that

$$\mathring{\Delta}(\aleph_n) = \max_i \sup\Big\{r: \ \exists x \text{ such that } x + \frac{r a_i^{1/d}}{(a_i f_i(x))^{1/d}} A \subset \mathring{C}_i \setminus \aleph_{N_i}^i\Big\},$$

and therefore

$$\mathring{\Delta}(\aleph_n) = \max_i \Big\{a_i^{1/d}\Delta(\aleph_{N_i}^i)\Big\} \text{ and } V(\aleph_n) = \max_i \Big\{a_i V(\aleph_{N_i}^i)\Big\}.$$

Now, in order to use Proposition 7 we need to see that the density is close to the uniform density on small squares and then apply Lemma 8. Let us observe that, for all $y \in C_i$,

$$\Big|f_i(y)|C_i| - 1\Big| = \Big|\frac{f(y)}{a_i}|C_i| - 1\Big| = \frac{1}{a_i}\Big|\int_{C_i} f(y)dt - \int_{C_i} f(t)dt\Big| \le \frac{1}{a_i} K \int_{C_i} |y-t|dt,$$

and as $|y - t| \le \sqrt{d}c_n$, we have that

$$\Big|f_i(y)|C_i| - 1\Big| \le \frac{1}{a_i} K\sqrt{d}c_n^{d+1} \le K_1 c_n \quad \forall y \in C_i,$$

where $K_1 = \sqrt{d}K/f_0$. We will apply now Lemma 8, (with $\varepsilon = K_1 c_n$). If we denote $N_i' = \lceil N_i(1 + 2K_1 c_n)\rceil$, $w' = w\frac{1-2K_1c_n}{1+K_1c_n}$ and $\mathcal{Y}_{N_i'}$ a sample of N_i' variables uniformly drawn on C_i it holds that

$$\mathbb{P}\left(N_i\Delta^d(\aleph_{N_i}) \le \frac{wN_i}{a_i n}\right) \le \mathbb{P}\left(N_i'\Delta^d(\mathcal{Y}_{N_i'}) \le \frac{w'N_i'}{a_i n}\right)\left(1 - \frac{1 + 2K_1c_n + n^{-1}}{(nK_1c_n)(1+K_1c_n)}\right)^{-1}.$$

In order to prove that $\mathbb{P}(U(\aleph_n) < x) \leq \exp(-\exp(-x))$ asymptotically, we have to prove that:

$$\left(1 - \frac{1 + 2K_1c_n + n^{-1}}{(nK_1c_n)(1 + K_1c_n)}\right)^{-m_n} \to 1, \tag{36}$$

and

$$\prod_{i=1}^{m_n} \mathbb{P}\left(N_i'\Delta^d(\mathcal{Y}_{N_i'}) \leq \frac{w'N_i'}{a_iN}\right) \to \exp(-\exp(-x)), \tag{37}$$

being $w = x + \log(n) + (d-1)\log(\log(n)) + \log(\alpha))$.

To see (36) observe that

$$\left(1 - \frac{1 + 2K_1c_n + n^{-1}}{(nK_1c_n)(1 + K_1c_n)}\right)^{-m_n} = \exp\left(\frac{m_n}{nK_1c_n}\right)(1 + o(1)), \tag{38}$$

where $m_n \leq Mc_n^{-d}$. Because $nc_n^{d+1} \to \infty$, the right hand side of (38) converge to 1, as desired.

Let us prove now (37). Observe that the product is similar to the one in equation (25) with N_i' instead of N_i and w' instead of w. Because $w' = w(1 + O(c_n))$, we have $w'^{d-1}\exp(-w') \sim w^{d-1}\exp(-w)$.

Then, if we denote by $\varepsilon' = \max_i \frac{|N_i' - a_in|}{na_i}$, equation (27) imply that to prove (37) it suffices to prove that there exist $\delta_n \to 0$ such that $\mathbb{P}(\log(n)\varepsilon' \geq \delta_n) \to 0$.

First, let us introduce $\varepsilon_i = \frac{|N_i - a_in|}{a_in}$, and $\varepsilon = \max|\varepsilon_i|$, then, proceeding in (29) we obtain

$$\begin{aligned}\mathbb{P}\Big(\varepsilon\log(n) \geq \log(n)^{-1}\Big) \leq& \frac{(\log(n))^4}{n}\sum_{i=1}^{m_n}\frac{1 - a_i}{a_i}\\ \leq& \frac{(\log(n))^4}{n}\sum_{i=1}^{m_n}\frac{1}{a_0}\\ \leq& \frac{(\log(n))^4}{n}\frac{Mc_n^{-d}}{c_n^d f_0}.\end{aligned}$$

Now, because $\varepsilon' \leq \max_i \frac{|N_i' - N_i|}{na_i} + \max_i \frac{|N_i - a_in|}{na_i} = \max_i \frac{|N_i' - N_i|}{na_i} + \varepsilon$ and $N_i' \leq N_i(1 + 2K_1c_nN_i) + 1$, it holds that $\varepsilon' \leq \max_i \frac{2K_1c_nN_i + 1}{na_i} + \varepsilon$. Because $1 + \varepsilon = N_i/(na_i)$ we have that $\varepsilon' \leq 2K_1c_n(1 + \varepsilon) + \frac{1}{f_0nc_n^d} + \varepsilon$. Therefore, for $\varepsilon \leq (\log(n))^{-2}$

$$\varepsilon' \leq 2K_1c_n(1 + \log(n)^{-2}) + \frac{1}{f_0nc_n^d} + (\log(n))^{-2}.$$

Thus,

$$\mathbb{P}\left(\log(n)\varepsilon' \geq 2K_1c_n\log(n)(1 + \log(n)^{-1}) + \frac{\log(n)}{f_0nc_n^d} + (\log(n))^{-1}\right) \leq \frac{(\log(n))^4M}{nc_n^{2d}f_0}.$$

Finally, because, $\log(n)c_n \to 0$, $\frac{\log(n)}{nc_n^d} \to 0$ and $\frac{(\log(n))^4}{nc_n^{2d}} \to 0$, we obtain $\delta_n = 2K_1 c_n \log(n)(1+\log(n)^{-1}) + \frac{\log(n)}{f_0 n c_n^d} + (\log(n))^{-1} \to 0$.

Second inequality We will do a sketch of the proof, the arguments are similar to those in the proof of Proposition 8 using Lemma 8 like in the first inequality.

As in Proposition 8 let us denote:

- $\rho_n = \frac{r_f \rho_A}{f_0^{1/d}} \left(\frac{\log(n)}{n}\right)^{1/d}$ with $\rho_A = \max_{x\in A} \|x\|$.
- $G = \cup_{i\neq j}^{m_n} (\overline{C_i} \cap \overline{C_j})$
- $H = S \setminus (\cup_i^{m_n} C_i)$, notice that $H \subset \partial S^{c_n}$.

Exactly as in the proof of Proposition 8 we have that

$$U(\aleph_n) \leq \max\left\{\mathring{U}(\aleph_n), U(\aleph_n, G^{\rho_n}), U(\aleph_n, H)\right\} \text{ eventually almost surely.}$$

The proof that $\mathbb{P}(\mathring{U}(\aleph_n) \leq x) \to \exp(-\exp(-x))$ is obtained exactly as we did to obtain the first inequality, but using equation (21). We will bound the covering numbers of G^{ρ_n} and H to apply Lemma 7, to conclude the proof of the second inequality.

G is the union of less than $m_n 2^d$ squares of size c_n let us call them D_i. Each of this squares D_i can be cover by less than $ac_n^{d-1}\rho_n^{-d+1}$ balls of radius ρ_n, centered at some points x_j^i. We have that $G^{\rho_n} \subset \bigcup_{i,j} \mathcal{B}(x_j^i, 2\rho_n)$ and each of this balls can be covered by $c^*\rho_n^d n$ balls of radius $n^{-1/d}$. So G can be covered by less than $m_n 2^d a c_n^{d-1} \rho_n^{-d+1} c^* \rho_n^d n$ balls, that is $2^d a c^* M c_n^{-1} \rho_n n$, which satisfies the conditions to apply Lemma 7 because $c_n = \mathcal{O}\big(\log(n)/n\big)^{1/3d}$.

In the same way it can be proved that H can be covered with $\mathcal{O}(nc_n)$ balls of radius $n^{-1/d}$, which satisfies the hypothesis to apply Lemma 7 for the proposed value of c_n.

□

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